Preparation and manipulation of a fault-tolerant superconducting qubit

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We describe a qubit encoded in continuous quantum variables of an rf superconducting quantum interference device. Since the number of accessible states in the system is infinite, we may protect its two-dimensional subspace from small errors introduced by the interaction with the environment and during manipulations. We show how to prepare the fault-tolerant state and manipulate the system. The discussed operations suffice to perform quantum computation on the encoded state, syndrome extraction, and quantum error correction. We also comment on the physical sources of errors and possible imperfections while manipulating the system.

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I. INTRODUCTION

Maintaining quantum coherence is crucial for quantum-information processing. Coherence of any quantum system is gradually suppressed due to unwanted interactions with the environment. Among proposed realization of qubits, solid-state devices appear particularly promising due to the scalability and ease of integration in electronic circuits, but their operation requires keeping them coherent, a potentially strong problem due to the host of microscopic modes. One of the achievements of quantum-information theory is the development of quantum-error-correction techniques, which allow one to suppress the effect of the environment on the software level, i.e. by running the appropriate quantum code to eliminate errors.[1] The standard approach protects against large errors that occur rarely.[2,3,4] Alternative methods were proposed by Gottesman and Kitaev [5], where the *shift-resistant* codes protect against errors that occur continuously but are weak. Specifically, they analyzed the situation when a qubit is embedded in an infinite-dimensional Hilbert space of a physical device with a continuous degree of freedom. Continuous-variable quantum codes have been further developed for both qubits and qudits (d-dimensional analog of qubits).[6,7,8,9,10] The continuous-variable codes became also a framework for discussion of quantum key distribution[11] or quantum teleportation of continuous quantum variables.[12].

In the original work,[5] the authors focused on encoding a qubit in an oscillator and developed codes which protect against small shifts in the canonical variables of the oscillator. They also discussed implementation of this approach in optical systems. A universal set of quantum logic gates and error-correction steps may be realized using linear optical operations, squeezing, homodyne detection, photon counting, and nonlinear couplings. While some of these steps are easily realized with optical means, the encoding and certain gates from the universal set, which require photon counting and nonlinear couplings, are difficult to implement (alternative method of encoded state preparation has been discussed in Ref. 13).

Here, we suggest the implementation of the shift-resistant codes in superconducting devices. We show that their physical properties simplify the implementation of the difficult steps mentioned above. In these systems one may use the charge[14,15,16] or the conjugate phase degree of freedom[17,18] to store and process quantum information. Typically, one adjusts the parameters such that the ground state is (almost) doubly degenerate, and at low energies the system reduces to a qubit. Here we consider a different approach, in which a continuous degree of freedom is used: the magnetic flux through the loop of an rf-SQUID (superconducting quantum interference device) (a superconducting loop interrupted by a Josephson junction) and the conjugate charge. The Josephson coupling can be tuned if one replaces the junction with a dc-SQUID.[19] For a turned-off Josephson coupling, the system behaves as a harmonic oscillator, and one can encode a qubit in its Hilbert space and manipulate the qubit's state. The specific useful features of this design are related to the periodic flux dependence of certain physical properties: this allows one to control the Josephson coupling and to monitor the magnetic flux modulo the flux quantum, thereby projecting out a comblike state needed for encoding of the qubit.

The goal of our work is twofold. On one hand, we suggest an implementation of certain steps needed for shift-resistant codes, which are hard to implement with optical means. On the other hand, one may think of implications of our results for the long-term strategy of quantum computing in superconducting systems. While fabrication of larger circuits should be relatively straightforward, it will be necessary to protect them against decoherence. Although the approach we discuss here imposes certain constraints on the system parameters and requires complicated operation procedures, the alternative of standard quantum-error-correction codes requires building circuits with many auxiliary qubits and performing complicated series of logic gates during error detec-

tion and recovery (cf. Ref. 20). Here, we demonstrate that the ideas of continuous-variable codes may be, in principle, implemented in superconducting circuits, underline advantages of these devices but do not optimize the design, and only superficially consider constraints on the circuit parameters.

We begin with a short description of the shift-resistant codes in Sec. II. In Sec. III, we briefly describe the physical system representing the qubit and discuss the proposed implementation of the difficult steps, i.e., of the encoding procedure and the quantum gates, including those needed for error correction. In Sec. IV, we discuss, for completeness, further necessary steps such as squeezing and translations in the phase space and two-qubit operations. We then comment on the error models relevant for practical devices and discuss constraints on the circuit parameters.

II. SHIFT-RESISTANT CODES FOR LC OSCILLATOR

While qubit is the simplest nontrivial quantum system, many physical systems offer the opportunity to utilize many levels and often a continuous spectrum to process quantum information. Gottesman and Kitaev [5] suggested to encode a logical qubit in a system with a continuous degree of freedom, an oscillator. They described error-correcting codes, which protect the state of the qubit against perturbations that cause weak diffusion of the position and momentum of the oscillator. Here, we briefly describe this proposal in the language of an LC circuit.

Consider an LC oscillator, with the Hamiltonian

$$H = \frac{\Phi^2}{2L} + \frac{Q^2}{2C} \,. \qquad (1)$$

Here, the dynamical variables are the magnetic flux Φ and the conjugate charge Q. For convenience, we use below the dimensionless flux and charge variables,

$$\Phi' = \frac{1}{\sqrt{\pi}}\left(\frac{C}{L}\right)^{1/4}\Phi\,, \quad Q' = \frac{1}{\sqrt{\pi}}\left(\frac{L}{C}\right)^{1/4}Q\,, \qquad (2)$$

and omit the primes. The rescaled variables satisfy the standard commutation relation

$$[\Phi, Q] = \frac{i}{\pi}.$$

In the proposal of Gottesman and Kitaev [5] the codewords (the states that encode the basis logic states of the qubit) are comblike superpositions, both in the flux and charge representations,

$$\begin{aligned} |\bar{0}\rangle &= \sum_{s=-\infty}^{\infty} |\Phi = 2s\rangle = \sum_{m=-\infty}^{\infty} |Q = m\rangle\,, \\ |\bar{1}\rangle &= \sum_{s=-\infty}^{\infty} |\Phi = 2s+1\rangle = \sum_{m=-\infty}^{\infty} (-1)^m |Q = m\rangle\,. \end{aligned} \qquad (3)$$

For the purpose of error correction one measures the value of Φ mod 1. Such a measurement provides information on the possible error, a shift of the comblike states [Eq. (3)], but does not distinguish between the combs peaked at even and odd flux values, $|\bar{0}\rangle$ and $|\bar{1}\rangle$. The observed shift is compensated by the inverse flux shift, which moves the comb structure to the closest integer. This procedure can correct sufficiently small flux-shift errors, $\Delta\Phi < 1/2$. Similarly, one can correct small shifts, $\Delta Q < 1/2$, in the charge variable Q. If the protection from rare large errors is also desired, these codes can be concatenated with the standard error-correction codes.[5]

The states [Eq. (3)] define a two-dimensional subspace protected from decoherence. In the following sections, we will show how one can prepare such states, store them (in the oscillatory regime with the Josephson coupling turned off), implement quantum logic gates and error-correction steps using the control over the Josephson coupling, the flux bias, and inductive coupling between qubits.

III. ENCODING THE QUBIT AND LOGIC GATES

A. Oscillator and the identity operation

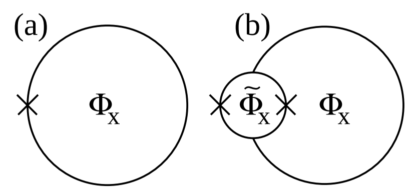

FIG. 1: The simplest flux qubits (Ref. 21) can be used to encode a qubit in their continuous variables. (a) The rf-SQUID, a simple loop with a Josephson junction, and (b) rf-SQUID with tunable Josephson coupling.

The Hamiltonian of an rf-SQUID with a tunable Josephson coupling [shown in Fig. 1(b)] reads

$$\begin{aligned} H &= \pi\omega_0\left[\frac{(\Phi-\Phi_{\rm x})^2}{2} + \frac{Q^2}{2}\right] \\ &\quad - E_J(\tilde{\Phi}_{\rm x})\cos\left[2\pi^{3/2}(L/C_J)^{1/4}\frac{\Phi}{\Phi_0}\right]\,. \end{aligned} \qquad (4)$$

Here, the first two terms, which form the Hamiltonian of an oscillator, are the magnetic energy, controlled via the external flux bias $\Phi_{\rm x}$ and the inductance L of the loop, and the charging energy (depending on the parameters the capacitance $C_{\rm J}$ may reduce to that of the junction or involve the geometry of the whole loop). $\omega_0 = (LC_J)^{-1/2}$ is the frequency of the LC oscillator. The last term is the Josephson energy, with magnitude

$$E_J(\tilde{\Phi}_{\rm x}) = 2E_J^0 \cos\left[\pi^{3/2}(L/C_{\rm J})^{1/4}\tilde{\Phi}_{\rm x}/\Phi_0\right]\,,$$

controlled by the flux $\tilde{\Phi}_{\mathrm{x}}$ through the small loop of the dc-SQUID.

We observe the qubit's states [Eq. (3)] in the interaction picture of dynamics. In this approach, the state of the qubit is conserved if no manipulations are performed (although in the Schrödinger representation, the Hamiltonian generates evolution), thus providing the identity operation. Specifically, consider the system in the oscillatory regime, when the Josephson coupling and the externally applied flux Φ_{x} are tuned to zero. Then for an arbitrary state, the Hamiltonian [Eq. (1)], with the parabolic potential centered at the origin, generates rotation (of the density matrix in the Wigner form) of the state with the frequency ω_0 in the phase space Φ-Q around the origin. In other words, the amplitudes Φ_R, Q_R in the rotating frame (interaction representation), which are related to Φ and Q as

$$\begin{pmatrix} \Phi_R \\ Q_R \end{pmatrix} = \begin{pmatrix} \cos\omega_0 t & -\sin\omega_0 t \\ \sin\omega_0 t & \cos\omega_0 t \end{pmatrix} \begin{pmatrix} \Phi \\ Q \end{pmatrix}, \qquad (5)$$

are integrals of motion, $\dot{\Phi}_R = \dot{Q}_R = 0$. Below we discuss quantum logic transformations in this reference frame. Note that performing operations in the rotating frame implies in practice that one has to keep track of the phase of the oscillator at frequency ω_0 and perform all steps relative to this reference signal.

B. Encoding

Using the possibility to measure the flux mod Φ_0, one can prepare a code state [Eq. (3)], $|\bar{0}\rangle$or $|\bar{1}\rangle$, by first preparing a state with a wide uniform distribution of flux and then measuring the flux value. As a result, one obtains a comblike state, with equidistant peaks and an offset of the structure, given by the result of the flux measurement. One performs a flux shift to compensate this offset and arrives at a codeword [Eq. (3)]. Possible methods to implement these steps are discussed below.

Preparation of a wide flux distribution may be realized by a number of methods. For instance, one may let the oscillator relax to the ground state in a narrow well, with the inductance decreased by (for instance, slowly) switching on a strong Josephson coupling. This creates a "squeezed" state, with a narrow flux distribution and a wide charge distribution. When the Josephson coupling is turned off, this distribution in the Φ-Q-plane starts rotating about the origin and after a quarter of the oscillator period, $\pi/(2\omega_0)$, a wide distribution of flux is reached, $\sim \int d\Phi\, |\Phi\rangle$. Alternatively, one can prepare such a state by squeezing the ground state of the oscillator (see below).

Now, to single out a comblike structure out of this wide distribution, one is to perform a measurement of the flux value up to a multiple of a certain offset. This can be achieved by coupling the rf-SQUID inductively to a readout dc-SQUID and reading out the critical current of the latter. This critical current is a periodic function of the total magnetic flux in the measuring SQUID's loop,

$$I_{\mathrm{c}} = 2I_{\mathrm{c0}} \cos\left[\pi \frac{\Phi_{\mathrm{mx}} + \lambda(\Phi - \Phi_{\mathrm{x}})}{\Phi_0}\right],$$

where Φ_{mx} is the external flux bias of the measuring SQUID and the coupling λ depends on the self- and mutual inductances of the loops, i.e., on the geometry of the SQUID's. For the particular geometry depicted in Fig. 2, the parameter λ can be close to 1 (our scheme is valid for different scenarios and values of λ as well).

Reading out the value of the critical current, one projects out a comblike state,

$$P_\alpha = \sum_{s=-\infty}^{\infty} |\Phi = s\Phi_{\mathrm{p}} + \alpha\rangle \langle \Phi = s\Phi_{\mathrm{p}} + \alpha|, \qquad (6)$$

with a period $\Phi_{\mathrm{p}} = \Phi_0/\lambda$ (if only the absolute value of critical current is measured and the sign is not resolved). α here is the initial displacement of the state from zero. (In fact, one would obtain a superposition of two comblike states with different offset values α since two series of delta peaks correspond to each value of the critical current; one method to leave only one of these is to read out the critical current again after a small shift of the flux bias Φ_{mx}.)

Starting from the state projected by Eq. (6) from the wide flux distribution, one could compensate for the shift α by performing the inverse flux shift and also change the period of the comb structure by squeezing (see the next section) to arrive at one of the basis states, $|\bar{0}\rangle$ or $|\bar{1}\rangle$. As we shall see later, for the purpose of performing one-qubit gates it is convenient to tune the peak separation to the superconducting flux quantum, Φ_0 (possibly by squeezing). However, according to the definition of the encoded states [Eq. (3)] and in the units defined in Eq. (2) the peak spacing in the state $|\bar{0}\rangle$ equals $2(\pi^2 L/C_J)^{1/4}$. Comparison to the flux quantum gives an extra constraint on the ratio L/C_J,[25]

$$\left(\frac{L}{C_J}\right)^{1/4} = \Phi_0/(2\sqrt{\pi}). \qquad (7)$$

With this condition satisfied, the Josephson term in the Hamiltonian of the system [Eq. (4)] becomes periodic in Φ with the period 2.

C. Quantum gates

If the Josephson coupling of the SQUID loop is nonzero, the Hamiltonian in the interaction picture has for $\Phi_x = 0$ the following form

$$H = -E_J(\tilde{\Phi}_{\mathrm{x}}) \cos\left[\pi\Phi_R \cos\omega_0 t + \pi Q_R \sin\omega_0 t\right]. \qquad (8)$$

For the times $t_z = k\pi/\omega_0$ and $t_x = \pi(k+1/2)/\omega_0$ (k being an integer), it reduces in the code subspace to

$$H(t_z) = -E_J(\tilde{\Phi}_{\mathrm{x}})\sigma_z, \qquad (9a)$$

$$H(t_x) = -E_J(\tilde{\Phi}_{\mathrm{x}})\sigma_x. \qquad (9b)$$

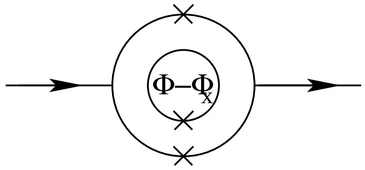

FIG. 2: A possible method of the inductive meter-qubit coupling. In this geometry (when the qubit is placed inside the meter), the coupling is strong, and $\lambda \approx 1$. For simplicity, the qubit is shown without the smaller, dc-SQUID loop.

Here, the Pauli matrices act on the encoded states $|\bar{0}\rangle$ and $|\bar{1}\rangle$ like on the spin states "up" and "down" along the z direction respectively. Using short pulses (the duration τ should satisfy $\tau\omega_0 \ll 1$) of the magnetic flux $\tilde{\Phi}_{\rm x}$ at the times $t_{x(z)}$, we can perform small phase shifts between the encoded states or induce transitions between them. Sufficiently large shifts, accumulated during many pulses, give rise to simplest one-qubit operations that generate the group of all possible unitary transformations in the encoded subspace.[22]

Certainly no operation can be performed instantaneously. So, apart from the instantaneous Hamiltonian corresponding exactly to the times $t = t_{x(z)}$, we should analyze the time dependence of the Hamiltonian near $t_{x(z)}$ to evaluate the error introduced by the finite rotation during the operation. Let us consider the Hamiltonian [Eq. (8)] at times close to t_z. Expansion to the quadratic terms in $\omega_0\tau$ gives

$$\begin{aligned} H \;\approx\; & -E_J(\tilde{\Phi}_{\rm x})\left\{\cos[\pi\Phi_R] - \pi Q_R \sin[\pi\Phi_R]\omega_0\tau - \right. \\ & \left. -[(\pi Q_R)^2\cos(\pi\Phi_R) - (\pi\Phi_R)\sin(\pi\Phi_R)](\omega_0\tau)^2\right\}. \end{aligned}$$

The error is generated by the time-dependent part. Since for the code space

$$\sin(\pi\Phi_R)|\bar{0}\rangle = \sin(\pi\Phi_R)|\bar{1}\rangle = 0,$$

the error is generated by at least second order terms in $\omega_0\tau$, and for $\omega_0\tau \ll 1$ it is small. In addition, as shown in Sec. V, errors generated by quadratic terms in Q can be corrected using the error-correcting routines. (The same property holds at times close to t_x.)

To complete the set of universal quantum gates, we need also a many-dimensional (in this case continuous) equivalent of CNOT, the SUM gate, which transforms the variables of two coupled qubits according to

$$\begin{aligned} \mathrm{SUM}: \quad & \Phi_1 \to \Phi_1, \quad Q_1 \to Q_1 - Q_2, \\ & \Phi_2 \to \Phi_1 + \Phi_2, \quad Q_2 \to Q_2. \end{aligned}$$

Here, the indices number the coupled qubits. This gate belongs to the so-called *symplectic group* and may be realized using (in quantum-optical setting) phase shifters, squeezing, and beam splitters[5,23] - elements that are accessible also for LC oscillators, as we will show in the next section. The SUM gate reduces in the code subspace to CNOT. Its continuous nature is, however, crucial during the syndrome extraction.

IV. MANIPULATING THE SYSTEM

So far, we presented the computational steps that are difficult with quantum-optical elements and much more natural with the Josephson-junction systems. To make the discussion complete, let us now discuss the operations that are necessary for the symplectic operations and can be performed on arbitrary wave function: squeezing, phase shifts, translations in the Φ-Q plane, and inductive coupling of two oscillators. In this section, all operations are described in the oscillatory regime $E_J(\tilde{\Phi}_{\rm x}) = 0$.

A. Squeezing and phase shifts

The scaling factors in Eq. (2) define dimensionless variables for which the Hamiltonian is parametrized by only one real parameter, the energy scale ω_0. Since the scaling depends on the ratio $L/C_{\rm J}$ and the frequency ω_0 on the product of L and $C_{\rm J}$, modification of either of the parameters (L or $C_{\rm J}$) influences the system behavior in two ways - the oscillation period changes and the evolution (rotation) path in the phase space is squeezed in one direction. In particular, suppose that there is an extra capacitance in parallel to the junction that can be switched instantaneously on and off, so that the total capacitance may equal C_J or λC_J. Switching at some instant of time [for simplicity we assume that this moment corresponds to $t = 0$ in Eq. (5)] from $C_{\rm J}$ to $\lambda C_{\rm J}$ rescales the variables, $\Phi = \Phi_R \to \lambda^{1/4}\Phi = \lambda^{1/4}\Phi_R, Q = Q_R \to \lambda^{-1/4}Q = \lambda^{-1/4}Q_R$, and modifies the frequency, $\omega_0 \to \omega_1 = \lambda^{-1/2}\omega_0$. Using this property, we can perform squeezing of arbitrary wave function. However, according to Eq. (7), there is a constraint relating the system parameters to the value of the flux quantum and we have to switch the capacitance back to the original value before performing further steps, like the single-qubit operations. The second switching (back from $\lambda C_{\rm J}$ to $C_{\rm J}$) should be performed after a time delay, corresponding to an exchange of the role of the variables. To be more specific suppose that we switch the capacitance from $C_{\rm J}$ to $\lambda C_{\rm J}$ at $t = 0$, switch it back at $t = \pi/2\omega_1$ (quarter of the full oscillation), and observe the effect after the period of oscillation is completed (then the laboratory and rotating frames coincide and the effect is identical, which simplifies the analysis). The evolution in the Heisenberg picture of dynamics (in the laboratory frame) is described by the inverse of Eq. (5),

$$\begin{pmatrix} \Phi \\ Q \end{pmatrix} = U^{-1}(\omega_{0(1)}, t)\begin{pmatrix} \Phi_R \\ Q_R \end{pmatrix},$$

where U^{-1} is the inverse of the rotation matrix in Eq. (5). In this picture (and frame), the scaling of variables corresponding to the switching from $C_{\rm J}$ to $\lambda C_{\rm J}$ is described by a diagonal matrix S with the elements $\lambda^{1/4}$ and $\lambda^{-1/4}$. Thus, the entire procedure is described by the following

operation:

$$\begin{aligned} U_{squeeze} &= U^{-1}(\omega_0, 3\pi/2\omega_0)S^{-1}U^{-1}(\omega_1, \pi/2\omega_1)S \\ &= \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}. \end{aligned}$$

The interaction-picture result is

$$U_{squeeze}\begin{pmatrix} \Phi_R \\ Q_R \end{pmatrix} = \begin{pmatrix} \lambda^{1/2}\Phi_R \\ \lambda^{-1/2}Q_R \end{pmatrix}.$$

This equation clearly describes squeezing of an arbitrary state. Note that after we change the capacitance of the qubit, the frequency is modified as well. This can be used in implementing another element – the shifter of the relative phase between two oscillators. If we do not want the state to be modified but only the evolution advanced (or delayed), we should switch the capacitance to λC_J for the full period $2\pi/\omega_1$.

B. Translations in the phase space

The evolution in the interaction picture (the rotating frame) is trivial, provided that the external flux Φ_x is zero or at least constant. Once we switch it to a finite value, the physical center of rotation in the phase space shifts by Φ_x, and to simplify the description again we would have to shift the frame as well. However, square pulses of the flux Φ_x give us the possibility to perform another useful and necessary in error-correcting routine transformation – translations in the phase space (in the Φ-Q plane).

To avoid these noninertial effects of the rotating frame, we describe this procedure in the laboratory frame (it lasts, however, for the full oscillation period, and like in the case of squeezing the effects are in both frames identical after the operation is over). Suppose that we begin with a state described by the wave function $\psi_0(\Phi)$ in the potential centered at the origin ($\Phi_x = 0$). At the time $t = 0$, we turn on the external flux $\Phi_x = -\beta/2$. After the time $t = \pi/\omega_0$, which corresponds to a half of the oscillation, the state is transformed into $\psi_1(\Phi) = \psi_0(-\Phi+\beta)$. Then, we turn the external flux off and after the oscillation is completed, we obtain $\psi_2(\Phi) = \psi_0(\Phi-\beta)$ – the initial state shifted by β in Φ. To shift the state in the Q direction we need to perform the same procedure on the Fourier-transformed state, i.e., to delay the initial moment of operation by $\pi/2\omega_0$.

C. Inductive coupling of oscillators

Two qubits may be coupled inductively using an additional LC circuit (see Fig. 3). The Hamiltonian of the LC oscillator and the qubits (for a moment we turn back to the natural units) is

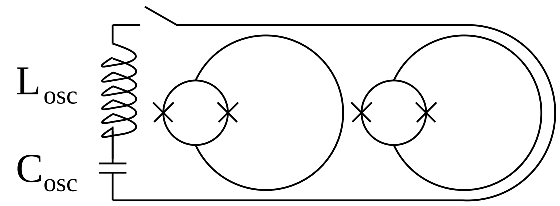

FIG. 3: Inductive coupling of two qubits is realized with the Josephson energy tuned to zero. It is thus the coupling of two LC oscillators. In the rotating frame of reference, this coupling provides equivalent of quantum-optical beam splitter (see the text for explanation).

$$\begin{aligned} H &= H_1 + H_2 + \Phi^2/2L_{osc} + Q^2/2C_{osc} - VQ, \\ V &= \sum_i M_i\dot{\Phi}_i/L, \end{aligned} \tag{10}$$

where H_1 and H_2 are oscillatorlike Hamiltonians of the form of Eq. (1), Q is the charge on the leads of the capacitor C_{osc}, and Φ is the flux through the external LC circuit. If the frequency of the oscillator is much bigger than the frequency of the qubits $\omega_{LC} = 1/\sqrt{L_{osc}C_{osc}} \gg \omega_0 = 1/\sqrt{LC_\mathrm{J}}$ (in the limit $C_{osc} \to 0$), we find that the coupling circuit remains in its ground state and the interaction (mediated by virtual excitations of the coupling circuit) is described by the term $-C_{osc}V^2/2$. Since $\dot{\Phi}_i = Q_i/C_J$, the interaction part of the Hamiltonian for two qubits has the form

$$H_{int} = -\sum_{i\neq j}\frac{C_{osc}M_iM_j}{L^2C_J^2}Q_iQ_j \equiv -K_{12}Q_1Q_2, \tag{11}$$

so that after integrating out the oscillator's variables, the Hamiltonian of the interacting system equals

$$H = H_1 + H_2 - K_{12}Q_1Q_2.$$

(Here, the parameters in $H_{1(2)}$ are slightly different than in Eq. (10) - from the term $-C_{osc}V^2/2$ we obtain the interaction part of the Hamiltonian (11) as well as terms quadratic in $Q_{1(2)}$, which formally slightly modifies the frequency of the qubits. In any case, this does not influence the general shape of our results - the phase shifts between groups of qubits can be compensated using the previously discussed techniques.) We can rewrite this Hamiltonian in the rotating frame of reference,

$$\begin{aligned} H \; &= -\frac{K_{12}}{2}\left[Q_{R1}Q_{R2}(1+\cos 2\omega_0 t) + \right. \\ & \left. \Phi_{R1}\Phi_{R2}(1-\cos 2\omega_0 t) + (Q_{R1}\Phi_2 + Q_{R2}\Phi_1)\sin 2\omega_0 t\right]. \end{aligned}$$

Since over time scales much longer than $1/\omega_0$ the effect of the oscillating terms averages out, we finally arrive at

$$H = -\frac{K_{12}}{2}\left(Q_{R1}Q_{R2} + \Phi_{R1}\Phi_{R2}\right).$$

Now, we solve the Heisenberg equations of motion for the operators with the initial conditions $\Phi_{1(2)}(0) = \Phi^0_{1(2)}$, $Q_{1(2)}(0) = Q^0_{1(2)}$ (for simplicity we omit here the R subscripts), and we arrive at

$$\begin{aligned} \Phi_{1(2)}(t) &= \Phi^0_{1(2)} \cos\frac{K_{12}t}{2} + Q^0_{2(1)} \sin\frac{K_{12}t}{2}, \\ Q_{1(2)}(t) &= -\Phi^0_{1(2)} \sin\frac{K_{12}t}{2} + Q^0_{2(1)} \cos\frac{K_{12}t}{2}. \end{aligned}$$

If we make further the substitution (which is done in practice by advancing the second oscillator by $\pi/2$),

$$\Phi_2 \to -Q_2, \quad Q_2 \to \Phi_2,$$

we obtain equations describing the action of a beam splitter[26] with time-dependent reflectivity and transmittance,

$$\sqrt{R} = \cos\frac{K_{12}t}{2}, \quad \sqrt{T} = \sin\frac{K_{12}t}{2}.$$

This element completes the set of all necessary operations.

V. PRACTICAL CONSIDERATIONS

A. Error models and error recovery

Errors that can be corrected using shift-resistant quantum codes are translations that leave the state out of the code subspace but still not too far from the initial state. The errors that can occur are due to the following interaction with the environment:

$$|\overline{x}\rangle_{qb}|0\rangle_E \to \left(e^{iQ\alpha}e^{i\Phi\beta}|\overline{x}\rangle\right)|\alpha,\beta\rangle_E. \tag{12}$$

Certainly, almost no physical interaction is of this kind, but it can be expanded in terms of such translations,

$$|\overline{x}\rangle_{qb}|0\rangle_E \to \int d\alpha' d\beta' C(\alpha',\beta',t)\left(e^{iQ\alpha'}e^{i\Phi\beta'}|\overline{x}\rangle\right)|\alpha',\beta'\rangle_E \tag{13}$$

The syndrome extraction procedure gives then one pair of real parameters (α, β) only, projecting the state onto Eq. (12). Effectiveness of the error recovery depends in this case on the amplitude C. We may treat the function $|C|^2$ as the probability distribution of possible errors (or equivalently C as two-dimensional wave function). If the uncertainties $\Delta\alpha'$ and $\Delta\beta'$ are small after the interval between two error-correcting routines, the states will be recovered with high fidelity.

The interaction with the environment can be also written in the operator-sum representation (more convenient for our purposes)

$$\rho(t) = \sum_i M_i \rho(0) M_i^\dagger, \tag{14}$$

$$M_i = \int d\alpha d\beta C_i(\alpha,\beta,t) e^{i\beta\Phi} e^{-i\alpha Q}. \tag{15}$$

In this case, if the error superoperator [acting on the qubit density matrix as in Eq. (14)] has support in operators that can be expanded in terms of small shifts the error correction will be effective.

This is indeed the case for some classes of physical errors. Amplitude damping of harmonic oscillator and a class of unitary errors like over-rotation or under-rotation have been described in Ref. 5. Such errors can occur easily from delay or advance in the pulse application, for instance during the single-qubit manipulations, where we need to keep track of the time evolution. However, the physical sources of errors can be different in quantum optical and superconducting systems. While in optics imperfections in operations lead mainly to amplitude damping (for instance absorption of light on the beam splitters and phase shifters), superconducting elements are dephased by the Josephson junctions. So, to make our discussion more explicit, we consider (apart from the models presented already in Ref. 5) also the phase damping of the LC oscillator.

Specifically, let us consider the full Hamiltonian of the rf-SQUID, where the external flux (Φ_x and $\tilde{\Phi}_{\mathrm{x}}$) may be slightly fluctuating

$$\begin{aligned} H &= -2E_J^0 \cos\left[\frac{\pi\tilde{\Phi}_{\mathrm{x}} + \pi\delta\tilde{\Phi}_{\mathrm{x}}(t)}{2}\right]\cos(\pi\Phi) \\ &+ \pi\omega_0\left[\frac{(\Phi - \Phi_x - \delta\Phi_x(t))^2}{2} + \frac{Q^2}{2}\right], \end{aligned}$$

where $\delta\tilde{\Phi}_{\mathrm{x}}(t)$ and $\delta\Phi_x(t)$ are the fluctuations. If we consider the oscillatory regime, i.e., $\tilde{\Phi}_{\mathrm{x}} = \pm 1$, we can expand the Hamiltonian around any integer value of $\Phi = k$,

$$\begin{aligned} H \approx\ & \mp E_J^0 \frac{\pi\delta\tilde{\Phi}_{\mathrm{x}}(t)}{2}(-1)^k\left(1 - \frac{\pi^2}{2}\Phi^2\right) + \\ & +\pi\omega_0\left[\frac{(\Phi-\Phi_x)^2}{2} + \frac{Q^2}{2}\right. \\ & \left. +(\Phi-\Phi_x)\delta\Phi_x(t) + \frac{(\delta\Phi_x(t))^2}{2}\right]. \end{aligned}$$

This Hamiltonian contains terms that describe coupling of the flux Φ to external flux fluctuations. Coupling of linear terms in Φ corresponds to the amplitude damping of the oscillator, while coupling of quadratic terms in Φ is associated with the phase damping. (Errors generated by quadratic terms in the variables can be also induced during the single-qubit operations, as discussed in Sec. III - they would also dephase the system.)

Let us start with the amplitude damping of the LC oscillator described by the following master equation:

$$\dot{\rho} = \Gamma\left(a\rho a^\dagger - \frac{1}{2}a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a\right).$$

Here,

$$a = \frac{1}{\sqrt{2}}(\Phi + iQ),$$
$$a^\dagger = \frac{1}{\sqrt{2}}(\Phi - iQ).$$

For short time intervals dt, we may write

$$\rho(t+dt) = \left(\sqrt{\Gamma dt}a\right)\rho(t)\left(\sqrt{\Gamma dt}a^\dagger\right) + \left(I - \frac{\Gamma}{2}a^\dagger a dt\right)\rho(t)\left(I - \frac{\Gamma}{2}a^\dagger a dt\right). \quad (16)$$

If we now compare Eqs. (15) and (16), we find the Kraus operators to be

$$M_1 = \sqrt{\Gamma dt}a, \quad M_2 = I - \frac{\Gamma}{2}a^\dagger a dt.$$

The amplitude corresponding to the first operator is[5]

$$C_1(\alpha,\beta,dt) = -\frac{i}{2}\left[\delta(\alpha)\delta(\beta - \sqrt{\Gamma dt/2}) - \delta(\alpha)\delta(\beta + \sqrt{\Gamma dt/2})\right] + \frac{1}{2}\left[\delta(\alpha - \sqrt{\Gamma dt/2})\delta(\beta) - \delta(\alpha + \sqrt{\Gamma dt/2})\delta(\beta)\right].$$

The second amplitude C_2 is found from

$$\int d\alpha d\beta C_2(\alpha,\beta,dt)e^{i\beta\Phi}e^{-i\alpha Q} = I - \frac{\Gamma dt}{2}a^\dagger a,$$

by applying inverse Fourier transform to both sides of this equation, which leads to

$$C_2(\alpha,\beta,dt) = \left(1 + \frac{\Gamma dt}{4}\right)\delta(\alpha)\delta(\beta) - \frac{\Gamma dt}{4}\left[\frac{\partial^2}{\partial\alpha^2}\delta(\alpha)\delta(\beta) + \delta(\alpha)\frac{\partial^2}{\partial\beta^2}\delta(\beta)\right].$$

Clearly, after short time intervals, the state can be only slightly shifted from the code subspace.

The amplitude damping of harmonic oscillator at finite temperatures leads to the thermal-state solution, i.e., $\rho(\infty) = e^{-\beta H}$. The thermal-state density matrix has all off-diagonal elements equal to 0. So, the amplitude damping gives rise indirectly also to dephasing. However, as already noted, in Josephson-junction systems it is worth analyzing the pure dephasing mechanism independently.

The latter process for the oscillator is described by the master equation

$$\dot\rho = \Gamma\left[a^\dagger a\rho a^\dagger a - \frac{1}{2}(a^\dagger a)^2\rho - \frac{1}{2}\rho(a^\dagger a)^2\right].$$

The same procedure like for amplitude damping gives

$$M_1 = \sqrt{\Gamma dt}a^\dagger a \approx \frac{1}{2i}\left(e^{i\sqrt{\Gamma dt}a^\dagger a} - e^{-i\sqrt{\Gamma dt}a^\dagger a}\right),$$
$$M_2 = I - \frac{\Gamma}{2}(a^\dagger a)^2 dt.$$

The first operator, M_1, is a sum of under-rotation and over-rotation of the oscillator and has been discussed in Ref. 5. M_2 is characterized by the amplitude

$$C_2(\alpha,\beta,dt) = \left(1 - \frac{\Gamma dt}{8}\right)\delta(\alpha)\delta(\beta) - \frac{\Gamma dt}{8}\left\{\frac{\partial^4}{\partial\alpha^4}\delta(\alpha)\delta(\beta) + \delta(\alpha)\frac{\partial^4}{\partial\beta^4}\delta(\beta) + \frac{\partial^2}{\partial\alpha^2}\delta(\alpha)\frac{\partial^2}{\partial\beta^2}\delta(\beta) - 2\left[\frac{\partial^2}{\partial\alpha^2}\delta(\alpha)\delta(\beta) + \delta(\alpha)\frac{\partial^2}{\partial\beta^2}\delta(\beta)\right]\right\},$$

which for small dt should be still sufficiently localized around the point $(0,0)$ to enable successful error correction.

Regardless of the analysis made, we could also argue that the amplitude C in Eq. (13) is, in some sense, a wave function. Initially, when the state is in the code subspace, $C(\alpha,\beta,t=0) = \delta(\alpha)\delta(\beta)$. Any kind of qubit-environment interaction generates evolution of the function and, if the process is physical, the wave function will smoothly spread in time. Thus, whatever the sources of errors, if the state's destruction is not too fast, we should be able reverse the effects of decoherence.

In the considered system, we may also reverse some effects of imperfections in manipulating the system. For instance, imprecise time measurement is equivalent to over-rotation or under-rotation, and imprecisely adjusted external flux Φ_x causes diffusion of the state in the Φ-Q plane.

Error recovery for this class of codes requires two steps corresponding to two independent errors: shifts in Φ and Q. The error syndrome (length of the shift) is measured on an ancillary qubit. First, we prepare the ancilla in the state

$$|\overline{0}\rangle + |\overline{1}\rangle = \sum_{s=-\infty}^{\infty}|\Phi = s\rangle.$$

If the state of the first qubit is shifted by α, we have the initial state of the data qubit and the ancilla in the form

$$|\overline{x+\alpha}\rangle(|\overline{0}\rangle + |\overline{1}\rangle).$$

Then, the SUM operation is performed. Since the gate acts on the states $|\overline{j}\rangle = \sum_s|\Phi = 2s + j\rangle$ like

$$\text{SUM:}|\overline{j}\rangle|\overline{k}\rangle \to |\overline{j}\rangle|\overline{j\oplus k}\rangle,$$

after the SUM operation, we arrive at

$$|\overline{x+\alpha}\rangle(|\overline{x+\alpha}\rangle + |\overline{x+1+\alpha}\rangle) = |\overline{x+\alpha}\rangle\left(\sum_{s=-\infty}^{\infty}|\Phi = s + x + \alpha\rangle\right).$$

The state of the ancilla is invariant under translation by 1, and we can omit x as it can be only 0 or 1. The state of the ancilla contains the information about the shift *only* and not about the actual state of the qubit. By

measuring (destructively) the state of the ancilla system, we read out α mod 1. If we then shift the state of the qubit by α, the error is corrected. The same procedure on the Fourier-transformed state (physically the operation needs to be delayed by $\pi/2\omega_0$ - after the time roles of Φ and Q are interchanged) yields the shift in Q that is to be corrected.

B. Requirements to the circuit parameters

The ingredients required to perform quantum computation and quantum-error-correction routines have been described here for a system which may be manipulated with great accuracy. However, in real physical systems, it is rarely the case. Before we conclude our discussion, let us itemize the main potential problems in manipulating the system.

First of all, we should briefly comment on the codewords. The states [Eq. (3)] are clearly not physical as they contain components with infinite energy and are non-normalizable. We need to replace them with approximate, physical codewords. In Ref. 5, the physical codewords are defined as superpositions of narrow Gaussians weighted by a Gaussian envelope. If the width of the narrow peaks [which substitute the delta functions in Eq. (3)] is κ and we want the states to be equally localized in Φ and Q, the width of the envelope should be κ^{-1}. In other words, the basis of the (realistic) code subspace can be defined as

$$|\tilde{j}\rangle \propto \sum_{s=-\infty}^{\infty} e^{-\pi^2\kappa^2(2s+j)^2/2} \int d\Phi e^{-\frac{1}{2}(\Phi-2s-j)^2/\kappa^2}|\Phi\rangle,$$

where $j \in \{0,1\}$. There is a finite overlap of the approximate codewords $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$, and the physical states can be confused during the logical-state readout. However, for a moderate expectation value of the number operator $\langle a^\dagger a\rangle \approx 16$ (a is the annihilation operator of the oscillator), the error probability is as low as 10^{-6} (the estimation of the error probability due to this overlap has been discussed in Ref. 5). If we require that the mean energy of the states is an order of magnitude smaller than the energy gap Δ (to eliminate dephasing caused by elementary excitations), we obtain the upper limit of the oscillator frequency roughly 1 GHz.

The most challenging in the realization part of the presented procedure may be encoding. The measurement is assumed to be close to perfect: its duration should be much shorter than the characteristic time scale of the oscillator $\tau_m \ll 1/\omega_0$. With present-day technology, magnetometers with a sensitivity of $10^{-5}\Phi_0/\mathrm{Hz}^{1/2}$ are available. If we want to obtain approximate codewords in the encoding procedure, we should limit the oscillator frequency even further, down to a few MHz, which makes the qubit quite vulnerable to external noise. (The dimensions of the system need to be increased. Moreover, as the oscillation period gets longer the errors accumulated during each oscillation become more severe.)

The next difficulty that appears here is the absolute time measurement. Once we have prepared the encoded state of the qubit, we need to keep track of the evolution with accuracy determined by $1/\omega_0$. This problem can be overcome for a reasonably small number of operations: even if the oscillator is slightly advanced (or delayed), we will reset the phase of the oscillator using error-correcting schemes. However, if the number of operations between the error-correcting subroutines is big, even small imperfections may shift the phase so that effective error recovery will be hardly achievable.

Finally, the assumed tiny peak separation in the encoded states (of the order of Φ_0), resulting in the constraint [Eq. (7)] might require very hardly achievable values of the capacitance and inductance of the LC oscillator. If we multiply the separation in flux by any odd integer, we arrive at much weaker conditions. The logical states $|\bar{0}\rangle$ and $|\bar{1}\rangle$ are still peaked at maxima and minima, respectively, and the transformation [Eq. (9a)] is still feasible. However, increased separation in Φ results in decreased separation of the peaks in the conjugate variable for which the second single-qubit gate [Eq. (9b)] does not work as intended. To overcome this difficulty, it would be necessary to combine the single-qubit gates with squeezing.

Certainly, if quantum computation is to be of practical meaning, after the calculations are over we need to read out the state of the register. Since the logical state is encoded in the flux states, using the same dc-SQUID as for the purpose of encoding, we may determine whether the state is closer to 0 or 1 (in other words, if the flux is closer to even or odd value). The potential problems in this procedure are of the same kind like those discussed above: we need to perform the measurement when the phase of the oscillator $\omega_0 t$ is very close to 0 (mod π) or $\pi/2$ (mod π) for the Fourier-transformed state. Also in this case, the time of the measurement should be much shorter than $1/\omega_0$.

VI. CONCLUDING COMMENTS

We described a qubit encoded in an infinite-dimensional system, the rf-SQUID. Utilizing the schemes for quantum error correction described in Ref. 5, we showed how to use the entire Hilbert space instead of using simply two states from the spectrum (as it has been proposed previously for the flux qubits), in this way enabling error-correcting routines with a single rf-SQUID. In principle, one can prepare the logical state and manipulate the system in a coherent manner so that universal computation and error correction are possible. The set of gates consists of symplectic operations, for which amplification of errors can be avoided,[5] and non-symplectic that can be realized in the system in much more natural way than in quantum-optical setting. We also discussed the

physical errors and possible difficulties in experimental realizations. From the latter, we see which procedures should be further optimized: the encoding accuracy, if independent of the time necessary to perform projective measurement, would not lead to very strong constrains on the characteristic time scale of the system.

Making use of the procedures presented here, we may consider also different schemes for quantum error correction in superconducting nanocircuits, which protect a state of a wave-packet in an entangled state of many oscillators (Ref. 24 and references therein). Apart from the problem of quantum fault-tolerant computation we may use the discussed operations to various, not strictly computational schemes, like unconditional quantum teleportation of the variables Φ, Q, i.e., not only of a two-dimensional subspace of the system but of its arbitrary wave function.[12]

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[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[2] D. Gottesman, Phys. Rev. A **54** , 1862 (1996).

[3] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane, Phys. Rev. Lett. **78**, 405 (1997).

[4] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane, IEEE Trans. Inf. Theory **44**, 1369 (1998).

[5] D. Gottesman, A. Kitaev, and J. Preskill, Phys. Rev. A **64**, 012310 (2001).

[6] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).

[7] S. D. Bartlett, B. C. Sanders, B. T. H. Varcoe, and H. de Guise, *Proceedings of IQC'01*, edited by R. G. Clark (Rinton, Princeton, NJ, 2001) pp. 344-347.

[8] S. Pirandola, S. Mancini, D. Vitali, and P. Tombesi, Europhys. Lett. **68**, 323 (2004).

[9] S. D. Bartlett, H. de Guise, and B. C. Sanders, Phys. Rev. A **65**, 052316 (2002).

[10] B. C. Sanders, S. D. Bartlett, and H. de Guise, Proceedings of the ICSSUR'01, edited by D. Han, Y. S. Kim, B. E. A. Saleh, A. V. Sergienko, and M. C. Teich, 2002 (unpublished).

[11] D. Gottesman and J. Preskill, Phys. Rev. A **63**, 022309 (2001).

[12] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).

[13] B. C. Travaglione and G. J. Milburn, Phys. Rev. A **66**, 052322 (2002).

[14] Y. Makhlin, G. Schoen, and A. Shnirman, Nature (London) **398**, 305 (1999).

[15] Yu. A. Pashkin, T. Yamamoto, O. Astafiev, Y. Nakamura, D. V. Averin, and J. S. Tsai, Nature (London) **421**, 823 (2003).

[16] T. Yamamoto, Yu. A. Pashkin, O. Astafiev, Y. Nakamura, and J. S. Tsai, Nature (London) **425**, 941 (2003).

[17] J. M. Martinis, S. Nam, J. Aumentado, and C. Urbina, Phys. Rev. Lett. **89**, 117901 (2002).

[18] K. B. Cooper, M. Steffen, R. McDermott, R. W. Simmonds, S. Oh, D. A. Hite, D. P. Pappas, and J. M. Martinis, Phys. Rev. Lett. **93**, 180401 (2004).

[19] M. Tinkham, *Introduction to Suerconductivity*, 2nd ed. (McGraw-Hill, New York, 1996).

[20] L.B. Ioffe, M. V. Feigel'man, A. Ioselevich, D. Ivanov, M. Troyer, and G. Blatter, Nature (London) **415**, 503 (2002).

[21] Y. Makhlin, G. Schön, and A. Shnirman, Rev. Mod. Phys. **73**, 357 (2001).

[22] A. Barenco, C.H. Bennett, R. Cleve, D.P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, Phys. Rev. A **52**, 3457 (1995).

[23] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).

[24] S. L. Braunstein, Nature (London) **394**, 47 (1998).

[25] The choice of units following from Eq.(2) makes the dimensionless variables fully symmetric under exchange in the Hamiltonian Eq.(1). It is thus the only one for which the oscillatory evolution is rotation around a circle (not ellipse) in the phase space. Conservation of the distance is crucial for the one-qubit operations and hence the constraint.

[26] In quantum optics the oscillator is simply a laser beam. The oscillatory variables q and p correspond to different quadratures of the electromagnetic field